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LETTER TO THE EDITOR

Maximal superintegrability on *N*-dimensional curved spaces

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Abstract

A unified algebraic construction of the classical Smorodinsky–Winternitz systems on the ND sphere, Euclidean and hyperbolic spaces through the Lie groups $SO(N+1)$, $ISO(N)$ and $SO(N,1)$ is presented. Firstly, general expressions for the Hamiltonian and its integrals of motion are given in a linear ambient space $\mathbb{R}^{N+1}$, and secondly they are expressed in terms of two geodesic coordinate systems on the ND spaces themselves, with an explicit dependence on the curvature as a parameter. On the sphere, the potential is interpreted as a superposition of $N+1$ oscillators. Furthermore, each Lie algebra generator provides an integral of motion and a set of $2N-1$ functionally independent ones are explicitly given. In this way the maximal superintegrability of the ND Euclidean Smorodinsky–Winternitz system is shown for any value of the curvature.

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Superintegrable systems on the two- and three-dimensional (3D) Euclidean spaces have been classified in [1, 2], and also extended to the 2D and 3D spheres [3] as well as to the hyperbolic spaces [4, 5]. Recent classifications of superintegrable systems for these 2D Riemannian spaces can be found in [6–8]. In the 2D sphere there are two (maximal) superintegrable potentials: the harmonic oscillator ($\tan^2 r$) with 'centrifugal terms' and the Kepler or Coulomb potential ($1/\tan r$) with some 'additional' terms. The former is the version with non-zero curvature of the Smorodinsky–Winternitz (SW) system [9–12]. Both potentials $\tan^2 r$ and $1/\tan r$ on the ND sphere have been studied in quantum mechanics in [13–15], and have been mutually related in [16, 17].

The SW Hamiltonian on the ND Euclidean space is given by

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + 2\beta_0 q_i^2 + \frac{2\beta_i}{q_i^2}\right). \qquad (1)$$

The following functions are integrals of motion for (1) ($i < j; i, j = 1, \ldots, N$):

$$I_{0i} = \tilde{P}_i^2 + 2\beta_0 q_i^2 + 2\frac{\beta_i}{q_i^2} \qquad \text{with} \quad \tilde{P}_i = p_i \tag{2}$$

$$I_{ij} = \tilde{J}_{ij}^2 + 2\beta_i \frac{q_j^2}{q_i^2} + 2\beta_j \frac{q_i^2}{q_j^2} \qquad \text{with} \quad \tilde{J}_{ij} = q_i p_j - q_j p_i. \tag{3}$$

Set (2) comes from the separability of the Hamiltonian $2\mathcal{H} = \sum_i I_{0i}$, while (3) are just the square of the components of the angular momentum tensor and some additional terms. The functions $\tilde{P}_i, \tilde{J}_{ij}$ close the commutation relations of the Euclidean algebra $iso(N)$ with respect to the canonical Lie–Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right). \tag{4}$$

Our aim is to construct, simultaneously, the non-zero curvature version of (1) on the three classical Riemannian spaces with constant curvature in arbitrary dimension, as well as to prove its maximal superintegrability, from a group theoretical standpoint.

Let $so_\kappa(N+1)$ be the real Lie algebra of the Lie group $SO_\kappa(N+1)$ with generators $\{J_{0i} \equiv P_i, J_{ij}\}$ ($i, j = 1, \ldots, N; i < j$) and non-vanishing commutation relations given by

$$\begin{array}{lll} [J_{ij}, J_{ik}] = J_{jk} & [J_{ij}, J_{jk}] = -J_{ik} & [J_{ik}, J_{jk}] = J_{ij} \\ [J_{ij}, P_i] = P_j & [J_{ij}, P_j] = -P_i & [P_i, P_j] = \kappa J_{ij} \end{array} \tag{5}$$

with $i < j < k$. If we consider the following Cartan decomposition of $so_\kappa(N+1)$:

$$so_\kappa(N+1) = h \oplus p \qquad h = \langle J_{ij} \rangle \quad p = \langle P_i \rangle \tag{6}$$

where h is the Lie algebra of $H \simeq SO(N)$, we obtain a family of ND symmetric homogeneous spaces $S^N_{[\kappa]} = SO_\kappa(N+1)/SO(N)$ parametrized by κ, which turns out to be the constant sectional curvature of the space. Thus, J_{ij} leave a point $\mathcal{O}$ invariant by acting as rotations, while P_i generate translations that move $\mathcal{O}$ along N basic geodesics l_i orthogonal at $\mathcal{O}$. For $\kappa >, =, < 0$, $S^N_{[\kappa]}$ reproduces the sphere $\mathbf{S}^N = SO(N+1)/SO(N)$, Euclidean $\mathbf{E}^N = ISO(N)/SO(N)$ and hyperbolic $\mathbf{H}^N = SO(N,1)/SO(N)$ spaces, respectively. The case $\kappa = 0$ is the contraction around $\mathcal{O}$: $\mathbf{S}^N \to \mathbf{E}^N \leftarrow \mathbf{H}^N$.

The vector representation of $so_\kappa(N+1)$ is given by $(N+1) \times (N+1)$ real matrices:

$$P_i = -\kappa e_{0i} + e_{i0} \qquad J_{ij} = -e_{ij} + e_{ji} \tag{7}$$

where e_{ij} is the matrix with entries $(e_{ij})^l_m = \delta^l_i \delta^m_j$. Any generator X of $so_\kappa(N+1)$ fulfils

$$X^{\mathrm{T}}\Lambda + \Lambda X = 0 \qquad \Lambda = e_{00} + \kappa \sum_{i=1}^{N} e_{ii} = \mathrm{diag}(1, \kappa, \ldots, \kappa) \tag{8}$$

so that any element $G \in SO_\kappa(N+1)$ verifies $G^{\mathrm{T}}\Lambda G = \Lambda$. In this way, $SO_\kappa(N+1)$ is a group of linear transformations in an ambient space $\mathbb{R}^{N+1}$, with Weierstrass coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$, acting as the group of isometries of the bilinear form Λ via matrix multiplication. The Lie group $H \simeq SO(N) = \langle J_{ij} \rangle$ is the isotopy subgroup of the origin $\mathcal{O} = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$. The space $S^N_{[\kappa]}$ is identified with the orbit of $\mathcal{O}$, which is contained in the 'sphere' Σ:

$$\Sigma \equiv x_0^2 + \kappa \sum_{i=1}^{N} x_i^2 = 1 \tag{9}$$

and the metric on $S^N_{[\kappa]}$ comes from the flat ambient metric in $\mathbb{R}^{N+1}$ in the form

$$\mathrm{d}s^2 = \frac{1}{\kappa}\left(\mathrm{d}x_0^2 + \kappa\sum_{i=1}^{N}\mathrm{d}x_i^2\right)\Bigg|_{\Sigma}. \tag{10}$$

A point $Q \in S^N_{[\kappa]}$ with Weierstrass coordinates $\mathbf{x}$ can be reached in different ways starting from $\mathcal{O}$ through the action of N one-parametric subgroups of $SO_\kappa(N+1)$:

$$\begin{aligned}\mathbf{x} &= \exp(a_1P_1)\exp(a_2P_2)\cdots\exp(a_{N-1}P_{N-1})\exp(a_NP_N)\mathcal{O}\\ &= \exp(\theta_N J_{N-1N})\exp(\theta_{N-1}J_{N-2N-1})\cdots\exp(\theta_2J_{12})\exp(rP_1)\mathcal{O}.\end{aligned} \tag{11}$$

The canonical parameters involved are intrinsic quantities on $S^N_{[\kappa]}$, called geodesic parallel $a = (a_1, \ldots, a_N)$ and geodesic polar $\theta = (r, \theta_2, \ldots, \theta_N)$ coordinates of the point $\mathbf{x}$:

$$\begin{aligned} x_0 &= \prod_{s=1}^{N}\mathrm{C}_\kappa(a_s) = \mathrm{C}_\kappa(r)\\ x_1 &= \mathrm{S}_\kappa(a_1)\prod_{s=2}^{N}\mathrm{C}_\kappa(a_s) = \mathrm{S}_\kappa(r)\cos\theta_2\\ x_i &= \mathrm{S}_\kappa(a_i)\prod_{s=i+1}^{N}\mathrm{C}_\kappa(a_s) = \mathrm{S}_\kappa(r)\prod_{s=2}^{i}\sin\theta_s\cos\theta_{i+1}\\ x_N &= \mathrm{S}_\kappa(a_N) = \mathrm{S}_\kappa(r)\prod_{s=2}^{N}\sin\theta_s \end{aligned} \tag{12}$$

where the curvature-dependent functions $\mathrm{C}_\kappa(x)$ and $\mathrm{S}_\kappa(x)$ are defined by [17, 18]:

$$\mathrm{C}_\kappa(x) = \begin{cases}\cos\sqrt{\kappa}x & \kappa > 0\\ 1 & \kappa = 0\\ \cosh\sqrt{-\kappa}x & \kappa < 0\end{cases} \qquad \mathrm{S}_\kappa(x) = \begin{cases}\frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}x & \kappa > 0\\ x & \kappa = 0.\\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}x & \kappa < 0\end{cases} \tag{13}$$

The κ-tangent is defined by $\mathrm{T}_\kappa(x) = \mathrm{S}_\kappa(x)/\mathrm{C}_\kappa(x)$; its contraction $\kappa = 0$ is $\mathrm{T}_0(x) = x$.

Each parallel coordinate a_i, associated with P_i, has dimensions of *length*: a_1 is the distance between $\mathcal{O}$ and a point Q_1, measured along the basic geodesic l_1; a_2 is the distance between Q_1 and another point Q_2, measured along a geodesic l_2' through Q_1 and orthogonal to l_1 (and 'parallel' in the sense of parallel transport to l_2) and so on, up to reaching Q [18]. On the other hand, the first polar coordinate r, associated with P_1, has dimensions of *length* and is the distance between $\mathcal{O}$ and Q measured along the geodesic l joining both points. The remaining θ_i, associated with J_{i-1i}, are ordinary *angles*, the polar angles of l relative to the reference flag at $\mathcal{O}$ spanned by $\{l_1\}, \{l_1, l_2\}, \ldots$. on the sphere $\mathbf{S}^N$ with positive curvature $\kappa = 1/R^2$, the usual spherical coordinates, all of which are angles, differ from ours [19] only in the first coordinate, which conventionally is taken as the dimensionless quantity r/R (see for instance [20]). When $\kappa = 0$, we recover directly the Cartesian and polar coordinates on $\mathbf{E}^N$.

Next, by introducing (12) in (10), we obtain the metric in $S^N_{[\kappa]}$:

$$\begin{aligned}\mathrm{d}s^2 &= \sum_{i=1}^{N-1}\left(\prod_{s=i+1}^{N}\mathrm{C}^2_\kappa(a_s)\right)\mathrm{d}a_i^2 + \mathrm{d}a_N^2\\ &= \mathrm{d}r^2 + \mathrm{S}^2_\kappa(r)\left(\mathrm{d}\theta_2^2 + \sum_{i=3}^{N}\left(\prod_{s=2}^{i-1}\sin^2\theta_s\right)\mathrm{d}\theta_i^2\right)\end{aligned} \tag{14}$$

which provides the kinetic energy $\mathcal{T}$ in terms of the velocities ($\dot{q} = \dot{a}, \dot{\theta}$), that is, the Lagrangian $\mathcal{L} \equiv \mathcal{T}$ of a geodesic motion on $S^N_{[\kappa]}$. If we introduce the canonical momenta $p = \partial\mathcal{L}/\partial\dot{q}$ ($p = p, \pi$), we obtain the free Hamiltonian $\mathcal{H} \equiv \mathcal{T}$ on $S^N_{[\kappa]}$:

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}\left(\sum_{i=1}^{N-1} \frac{p_i^2}{\prod_{s=i+1}^{N} \mathrm{C}_\kappa^2(a_s)} + p_N^2\right)\\ &= \frac{1}{2}\left(\pi_1^2 + \frac{\pi_2^2}{\mathrm{S}_\kappa^2(r)} + \sum_{i=3}^{N} \frac{\pi_i^2}{\mathrm{S}_\kappa^2(r)\prod_{s=2}^{i-1}\sin^2\theta_s}\right).\end{aligned} \tag{15}$$

An N-particle realization of $so_\kappa(N+1)$ in the phase space is obtained by starting from the following expressions in terms of Weierstrass coordinates:

$$\tilde{P}_i(x(q), \dot{x}(q,p)) = x_0\dot{x}_i - x_i\dot{x}_0 \qquad \tilde{J}_{ij}(x(q), \dot{x}(q,p)) = x_i\dot{x}_j - x_j\dot{x}_i \tag{16}$$

and expressing everything either in parallel (a, p) or polar (θ, π) canonical coordinates and momenta. In geodesic parallel coordinates, we obtain that $(i, j = 1, \ldots, N)$

$$\begin{aligned}\tilde{P}_i &= \prod_{k=1}^{i} \mathrm{C}_\kappa(a_k)\mathrm{C}_\kappa(a_i)p_i + \kappa\mathrm{S}_\kappa(a_i)\sum_{s=1}^{i}\mathrm{S}_\kappa(a_s)\frac{\prod_{m=1}^{s}\mathrm{C}_\kappa(a_m)}{\prod_{l=s}^{i}\mathrm{C}_\kappa(a_l)}p_s\\ \tilde{J}_{ij} &= \mathrm{S}_\kappa(a_i)\mathrm{C}_\kappa(a_j)\prod_{s=i+1}^{j}\mathrm{C}_\kappa(a_s)p_j - \frac{\mathrm{C}_\kappa(a_i)\mathrm{S}_\kappa(a_j)}{\prod_{k=i+1}^{j}\mathrm{C}_\kappa(a_k)}p_i\\ &\quad + \kappa\mathrm{S}_\kappa(a_i)\mathrm{S}_\kappa(a_j)\sum_{s=i+1}^{j}\mathrm{S}_\kappa(a_s)\frac{\prod_{m=i+1}^{s}\mathrm{C}_\kappa(a_m)}{\prod_{l=s}^{j}\mathrm{C}_\kappa(a_l)}p_s\end{aligned} \tag{17}$$

while in geodesic polar coordinates the same quantities read $(i, j = 1, \ldots, N-1)$

$$\begin{aligned}\tilde{P}_i &= \frac{\prod_{k=2}^{i+1}\sin\theta_k}{\tan\theta_{i+1}}\pi_1 + \sum_{s=2}^{i+1}\frac{\prod_{m=s}^{i+1}\sin\theta_m\cos\theta_s\pi_s}{\mathrm{T}_\kappa(r)\tan\theta_{i+1}\prod_{l=2}^{s}\sin\theta_l} - \frac{\pi_{i+1}}{\mathrm{T}_\kappa(r)\prod_{l=2}^{i+1}\sin\theta_l}\\ \tilde{P}_N &= \prod_{k=2}^{N}\sin\theta_k\pi_1 + \sum_{s=2}^{N}\frac{\prod_{m=s}^{N}\sin\theta_m\cos\theta_s}{\mathrm{T}_\kappa(r)\prod_{l=2}^{s}\sin\theta_l}\pi_s\\ \tilde{J}_{ij} &= \sin\theta_{i+1}\cos\theta_{j+1}\prod_{k=i+1}^{j}\sin\theta_k\pi_{i+1} - \frac{\cos\theta_{i+1}\sin\theta_{j+1}}{\prod_{l=i+1}^{j}\sin\theta_l}\pi_{j+1}\\ &\quad + \cos\theta_{i+1}\cos\theta_{j+1}\sum_{s=i+1}^{j}\frac{\prod_{m=s}^{j}\sin\theta_m\cos\theta_s}{\prod_{l=i+1}^{s}\sin\theta_l}\pi_s\\ \tilde{J}_{iN} &= \sin\theta_{i+1}\prod_{k=i+1}^{N}\sin\theta_k\pi_{i+1} + \cos\theta_{i+1}\sum_{s=i+1}^{N}\frac{\prod_{m=s}^{N}\sin\theta_m\cos\theta_s}{\prod_{l=i+1}^{s}\sin\theta_l}\pi_s.\end{aligned} \tag{18}$$

Both sets of generators (17) and (18) fulfil the commutation rules (5) with respect to the canonical Poisson bracket. The kinetic energy is related to the second-order Casimir of $so_\kappa(N+1)$ through

$$2\mathcal{T} = \tilde{\mathcal{C}} = \sum_{i=1}^{N}\tilde{P}_i^2 + \kappa\sum_{i,j=1}^{N}\tilde{J}_{ij}^2 \tag{19}$$

so that any generator Poisson-commutes with $\mathcal{T}$. The geodesic motion is maximally superintegrable and its integrals of motion come from any function of the Lie generators.

Now the crucial problem is to find potentials $\mathcal{U}(q)$ that can be added to $\mathcal{T}$ in such a manner that the new Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ preserves the maximal superintegrability. This requires adding 'some' terms to 'some' functions of the generators in order to ensure their involutivity with respect to $\mathcal{H}$. By taking into account the results given in [6] for $\mathbf{S}^2$ and $\mathbf{H}^2$, we propose the following generalization of the SW potential (1) to the space $S^N_{[\kappa]}$:

$$\begin{aligned}\mathcal{U} &= \beta_0 \frac{\sum_{s=1}^N x_s^2}{x_0^2} + \sum_{i=1}^N \frac{\beta_i}{x_i^2} = \beta_0 \sum_{i=1}^N \frac{\mathrm{S}_\kappa^2(a_i)}{\prod_{s=1}^i \mathrm{C}_\kappa^2(a_s)} \\ &\quad + \sum_{i=1}^{N-1} \frac{\beta_i}{\mathrm{S}_\kappa^2(a_i)\prod_{s=i+1}^N \mathrm{C}_\kappa^2(a_s)} + \frac{\beta_N}{\mathrm{S}_\kappa^2(a_N)} = \beta_0 T_\kappa^2(r) \\ &\quad + \frac{1}{\mathrm{S}_\kappa^2(r)}\left(\frac{\beta_1}{\cos^2\theta_2} + \sum_{i=2}^{N-1}\frac{\beta_i}{\cos^2\theta_{i+1}\prod_{s=2}^i \sin^2\theta_s} + \frac{\beta_N}{\prod_{s=2}^N \sin^2\theta_s}\right).\end{aligned} \tag{20}$$

On the sphere $\mathbf{S}^N$ with $\kappa > 0$, this can be interpreted as the joint potential due to a superposition of $N+1$ harmonic oscillators whose centres are placed at $N+1$ points on $\mathbf{S}^N$ mutually separated a quadrant (a distance $\pi/2\sqrt{\kappa}$, which for $\kappa = 1$ is $\pi/2$); on $\mathbf{S}^2$ these would be placed at the three vertices of an sphere's octant [21]. Explicitly, if we take $\kappa = 1$ and consider the polar coordinate r together with N geodesic distances r_i $(i = 1, \ldots, N)$ such that $x_0 = \cos r$, $x_i = \cos r_i$, the potential (20) turns out to be

$$\mathcal{U} = \beta_0 \tan^2 r + \sum_{i=1}^N \frac{\beta_i}{\cos^2 r_i} = \beta_0 \tan^2 r + \sum_{i=1}^N \beta_i \tan^2 r_i + \sum_{i=1}^N \beta_i. \tag{21}$$

The first term is $\beta_0 \tan^2 r$, where r is the distance from the particle and the origin $\mathcal{O}$ along the geodesic l; this is the spherical Higgs potential with centre at $\mathcal{O}$ where the 0th coordinate axis x_0 in the ambient space intersects the sphere. Each of the N remaining terms (apparently very different in (20)), $\beta_i \tan^2 r_i$, is written in terms of the spherical distance r_i to the point where the ith coordinate axis x_i intersects the sphere. Under the contraction $\kappa = 0, \mathbf{S}^N \to \mathbf{E}^N$, the first term gives rise to the 'flat' harmonic oscillator $r^2 = \sum_i a_i^2$, while the N remaining oscillators (whose centres would be now 'at infinity') leave the 'centrifugal' barriers β_i / a_i^2 as their imprints.

Let us consider the following functions I_{ij} $(i < j;\, i, j = 0, 1, \ldots, N)$:

$$I_{ij} = (x_i \dot{x}_j - x_j \dot{x}_i)^2 + 2\beta_i \frac{x_j^2}{x_i^2} + 2\beta_j \frac{x_i^2}{x_j^2} \tag{22}$$

which are quadratic in the momenta through the square of the generators. In parallel coordinates with the phase-space realization (17), they turn out to be

$$\begin{aligned} I_{0i} &= \tilde{P}_i^2 + 2\beta_0 \frac{\mathrm{S}_\kappa^2(a_i)}{\prod_{s=1}^i \mathrm{C}_\kappa^2(a_s)} + 2\beta_i \frac{\prod_{s=1}^i \mathrm{C}_\kappa^2(a_s)}{\mathrm{S}_\kappa^2(a_i)} \\ I_{ij} &= \tilde{J}_{ij}^2 + 2\beta_i \frac{\mathrm{S}_\kappa^2(a_j)}{\mathrm{S}_\kappa^2(a_i)\prod_{s=i+1}^j \mathrm{C}_\kappa^2(a_s)} + 2\beta_j \frac{\mathrm{S}_\kappa^2(a_i)\prod_{s=i+1}^j \mathrm{C}_\kappa^2(a_s)}{\mathrm{S}_\kappa^2(a_j)}. \end{aligned} \tag{23}$$

Likewise these can be written in geodesic polar coordinates. Hereafter, we consider the Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ with $\mathcal{T}$ and $\mathcal{U}$ given in (15) and (20). Note that the analogous property to (19) is given by

$$2\mathcal{H} = \sum_{i=1}^N I_{0i} + \kappa \sum_{i,j=1}^N I_{ij} + 2\kappa \sum_{i=1}^N \beta_i. \tag{24}$$

When $\kappa = 0$, the expressions (17), (23) and (24) reduce to (1)–(3). Next it can be proven that:

Proposition 1. *The $N(N+1)/2$ functions (23) are integrals of the motion for $\mathcal{H}$.*

Let us choose the following subsets $Q^{(k)}$ and $Q_{(k)}$ of $N-1$ integrals ($k = 2, \ldots, N$):

$$Q^{(k)} = \sum_{i,j=1}^{k} I_{ij} \qquad Q_{(k)} = \sum_{i,j=N-k+1}^{N} I_{ij} \tag{25}$$

where $Q^{(N)} \equiv Q_{(N)}$. The maximal superintegrability of $\mathcal{H}$ is characterized as follows.

Theorem 2.

(i) The N functions $\{Q^{(2)}, \ldots, Q^{(N)}, \mathcal{H}\}$ are mutually in involution. The same property holds for the set $\{Q_{(2)}, \ldots, Q_{(N)}, \mathcal{H}\}$.

(ii) The $2N-1$ functions $\{Q^{(2)}, \ldots, Q^{(N-1)}, Q^{(N)} \equiv Q_{(N)}, Q_{(N-1)}, \ldots, Q_{(2)}, I_{0i}, \mathcal{H}\}$ (with i fixed) are functionally independent, thus $\mathcal{H}$ is maximally superintegrable.

The set $Q^{(k)}$ can be associated with a sequence of orthogonal subalgebras within $h = so(N) = \langle J_{ij} \rangle$, the generators of which determine the terms quadratic in the momenta in the integrals I_{ij} starting 'upwards' from $\langle J_{12} \rangle = so(2)$:

$$\begin{array}{cccccccc} Q^{(2)} & \subset Q^{(3)} & \subset \ldots & \subset Q^{(k)} & \subset \ldots & \subset Q^{(N-1)} & \subset Q^{(N)} \\ so(2) & \subset so(3) & \subset \ldots & \subset so(k) & \subset \ldots & \subset so(N-1) & \subset so(N) \end{array}$$

with a similar embedding for $Q_{(k)}$ but starting 'backwards' from $\langle J_{N-1\,N} \rangle = so(2)$. In fact, the SW system on $\mathbf{E}^N$ can be constructed from a coalgebra approach [22] by means of N copies of $sl(2, \mathbb{R})$. When $\kappa = 0$, each $Q^{(k)}$ (or $Q_{(k)}$) is related to the kth order coproduct of the Casimir of $sl(2, \mathbb{R})$ [23]. In this sense, the results of theorem 2 show that the set of integrals ensuring the maximal superintegrability of the 'flat' SW system coming from a $sl(2, \mathbb{R})$-coalgebra also hold for any curvature.

Explicit proofs and details for this algebraic construction (which could also be applied to the ND Kepler potential) will be given elsewhere. Furthermore, the consideration of a second contraction parameter κ_2, that determines the signature of the metric [17, 18], would allow one to obtain superintegrable systems on different spacetimes.

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References

[1] Evans N W 1990 *Phys. Rev.* A **41** 5666
[2] Kalnins E G, Williams G C, Miller W Jr and Pogosyan G S 1999 *J. Math. Phys.* **40** 708
[3] Grosche C, Pogosyan G S and Sissakian A N 1995 *Fortschr. Phys.* **43** 523
[4] Kalnins E G, Miller W Jr and Pogosyan G S 1997 *J. Math. Phys.* **38** 5416
[5] Grosche C, Pogosyan G S and Sissakian A N 1997 *Phys. Part. Nuclei* **28** 486
[6] Rañada M F and Santander M 1999 *J. Math. Phys.* **40** 5026
[7] Kalnins E G, Miller W Jr and Pogosyan G S 2000 *J. Phys. A: Math. Gen.* **33** 6791
[8] Kalnins E G, Kress J M, Pogosyan G S and Miller W Jr 2001 *J. Phys. A: Math. Gen.* **34** 4705
[9] Fris J, Mandrosov V, Smorodinsky Ya A, Uhlir M and Winternitz P 1965 *Phys. Lett.* **16** 354

[10] Evans N W 1990 *Phys. Lett.* A **147** 483

[11] Evans N W 1991 *J. Math. Phys.* **32** 3369

[12] Grosche C, Pogosyan G S and Sissakian A N 1995 *Fortschr. Phys.* **43** 453

[13] Schrödinger E 1940 *Proc. R. Ir. Acad.* A **46** 9

[14] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309

[15] Leemon H I 1979 *J. Phys. A: Math. Gen.* **12** 489

[16] Kalnins E G, Miller W Jr and Pogosyan G S 2002 *Phys. Atom. Nucl.* **65** 1086
Nersessian A and Pogosyan G 2001 *Phys. Rev.* A **63** 020103(R)

[17] Herranz F J, Ortega R and Santander M 2000 *J. Phys. A: Math. Gen.* **33** 4525

[18] Herranz F J and Santander M 2002 *J. Phys. A: Math. Gen.* **35** 6601

[19] Rañada M F and Santander M 2002 *J. Math. Phys.* **43** 431

[20] Izmest'ev A A, Pogosyan G S, Sissakian A N and Winternitz P 1999 *J. Math. Phys.* **40** 1549

[21] Rañada M F, Santander M and Sanz-Gil T 2002 *Superintegrable Potentials and the Superposition of Higgs Oscillators on the Sphere* S^2 (Warszawa: Banach Center Publications) at press

[22] Ballesteros A and Ragnisco O 1998 *J. Phys. A: Math. Gen.* **31** 3791

[23] Ballesteros A and Herranz F J 1999 *J. Phys. A: Math. Gen.* **32** 8851